The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

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Topic Generator - Solution Set
Solutions

1. The value of $(\frac{4}{5})(\frac{5}{6})(\frac{6}{7})(\frac{7}{8})(\frac{8}{9})$ is
(A) $\frac{4}{9}$ (B) 1 (C) $\frac{6}{7}$ (D) 36 (E) $\frac{36}{25}$

Source: 2005 Cayley Grade 10 #2

Primary Topics: Number Sense

Secondary Topics: Operations | Fractions/Ratios

Answer: A

Solution:

Cancelling common factors in the numerators and denominators,

$$\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)\left(\frac{6}{7}\right)\left(\frac{7}{8}\right)\left(\frac{8}{9}\right) = \left(\frac{4}{\not{5}}\right)\left(\frac{\not{5}}{\not{6}}\right)\left(\frac{\not{6}}{\not{7}}\right)\left(\frac{\not{7}}{\not{8}}\right)\left(\frac{\not{8}}{9}\right) = \frac{4}{9}$$

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2. If $\frac{1}{3}x = 12$, then $\frac{1}{4}x$ equals
(A) 1 (B) 16 (C) 9 (D) 144 (E) 64

Source: 2005 Pascal Grade 9 #7

Primary Topics: Algebra and Equations

Secondary Topics: Equations Solving | Fractions/Ratios

Answer: C

Solution:

Solution 1
Since $\frac{1}{3}x = 12$, then $x = 3 \times 12 = 36$, so $\frac{1}{4}x = \frac{1}{4}(36) = 9$.

Solution 2
Since $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$, then $\frac{1}{4}x = \frac{3}{4} \times (\frac{1}{3}x) = \frac{3}{4}(12) = 9$.

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3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ is equal to
   (A) 1 (B) $\frac{1}{64}$ (C) $\frac{3}{14}$ (D) $\frac{7}{8}$ (E) $\frac{3}{8}$

Source: 2008 Gauss Grade 7 #3

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios

Answer: D

Solution:

Using a common denominator of 8, we have $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$.

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4. Rounded to 2 decimal places, $\frac{7}{9}$ is
   (A) 0.7 (B) 0.77 (C) 0.78 (D) 0.79 (E) 0.8

Source: 2009 Gauss Grade 8 #5

Primary Topics: Number Sense

Secondary Topics: Decimals | Fractions/Ratios

Answer: C

Solution:

Calculating, $\frac{7}{9} = 7 \div 9 = 0.7777\cdots = 0.\overline{7}$. Rounded to 2 decimal places, $\frac{7}{9}$ is 0.78.

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5. A recipe calls for $4\frac{1}{2}$ cups of flour. If you only make half of the recipe, then how many cups of flour do you need?
   (A) $2\frac{1}{2}$ (B) $2\frac{1}{4}$ (C) 9 (D) 2 (E) $2\frac{3}{4}$

Source: 2011 Gauss Grade 8 #7

Primary Topics: Algebra and Equations

Secondary Topics: Fractions/Ratios

Answer: B

Solution:

Solution 1
To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
Since half of 4 is 2 and half of $\frac{1}{2}$ is $\frac{1}{4}$, then $2\frac{1}{4}$ cups of flour are needed.

Solution 2
To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
To determine half of $4\frac{1}{2}$, we divide $4\frac{1}{2}$ by 2, or multiply by $\frac{1}{2}$.
Thus, $4\frac{1}{2} \times \frac{1}{2} = \frac{9}{2} \times \frac{1}{2} = \frac{9}{4} = 2\frac{1}{4}$ cups of flour are needed to make half of the recipe.

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6. Which of these fractions is larger than $\frac{1}{2}$?
   (A) $\frac{2}{5}$ (B) $\frac{3}{7}$ (C) $\frac{4}{7}$ (D) $\frac{3}{8}$ (E) $\frac{4}{9}$

Source: 2016 Gauss Grade 7 #4

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios

Answer: C

Solution:

A positive fraction is larger than $\frac{1}{2}$ if its denominator is less than two times its numerator.
Of the answers given, $\frac{4}{7}$ is the only fraction in which the denominator, 7, is less than 2 times its numerator, 4 (since $2 \times 4 = 8$).
Therefore, $\frac{4}{7}$ is larger than $\frac{1}{2}$.

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7. A box of fruit contains 20 apples, 10 oranges, and no other fruit. When a fruit is randomly chosen from the box, what is the probability that the fruit is an orange?
   (A) $\frac{1}{10}$ (B) $\frac{1}{20}$ (C) $\frac{1}{30}$ (D) $\frac{1}{3}$ (E) $\frac{2}{3}$

Source: 2016 Gauss Grade 7 #7

Primary Topics: Counting and Probability

Secondary Topics: Probability | Fractions/Ratios

Answer: D

Solution:

Each of the 30 pieces of fruit in the box is equally likely to be chosen. Since there are 10 oranges in the box, then the probability that the chosen fruit is an orange is $\frac{10}{30}$ or $\frac{1}{3}$.

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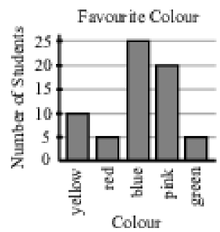
8. Students were surveyed about their favourite colour and the results are displayed in the graph shown. What is the ratio of the number of students who chose pink to the number of students who chose blue?



(A) 4 : 5 (B) 3 : 5 (C) 1 : 5 (D) 2 : 5 (E) 5 : 3

Source: 2017 Gauss Grade 8 #7

Primary Topics: Data Analysis

Secondary Topics: Graphs | Fractions/Ratios

Answer: A

Solution:

Reading from the graph, 20 students chose pink and 25 students chose blue.

The ratio of the number of students who chose pink to the number of students who chose blue is 20 : 25.

After simplifying this ratio (dividing each number by 5), 20 : 25 is equal to 4 : 5.

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9. Matilda and Ellie divide a white wall in their bedroom in half, each taking half of the wall. Matilda paints half of her section red. Ellie paints one third of her section red. The fraction of the entire wall that is painted red is

(A) $\frac{5}{12}$ (B) $\frac{2}{5}$ (C) $\frac{2}{3}$ (D) $\frac{1}{6}$ (E) $\frac{1}{2}$

Source: 2019 Cayley Grade 10 #10

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios

Answer: A

Solution:

Matilda and Ellie each take $\frac{1}{2}$ of the wall.
Matilda paints $\frac{1}{2}$ of her half, or $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the entire wall.
Ellie paints $\frac{1}{3}$ of her half, or $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the entire wall.
Therefore, $\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$ of the wall is painted red.

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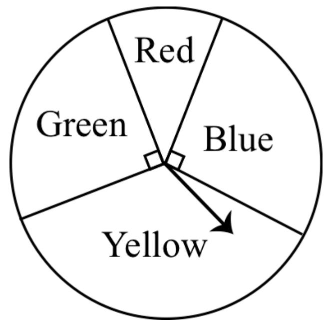
10. A circular spinner is divided into 4 sections, as shown. The angles at the centre of the circle in the sections labelled Green and Blue each measure 90°.



An arrow is attached to the centre of the spinner. The arrow is spun once. What is the probability that the arrow lands on either Red or Yellow?

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2}$ (E) $\frac{3}{4}$

Source: 2022 Pascal Grade 9 #8

Primary Topics: Data Analysis | Counting and Probability

Secondary Topics: Circles | Percentages | Fractions/Ratios

Answer: D

Solution:

The total central angle in a circle is 360°.
Since the Green section has an angle at the centre of the circle of 90°, this section corresponds to $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of the circle.
This means that when the spinner is spun once, the probability that it lands on the Green section is $\frac{1}{4}$.
Similarly, the probability that the spinner lands on Blue is also $\frac{1}{4}$.
Since the spinner lands on one of the four colours, the probability that the spinner lands on either Red or Yellow is $1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$.

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11. Harry charges \$4 to babysit for the first hour. For each additional hour, he charges 50% more than he did for the previous hour. How much money in total would Harry earn for 4 hours of babysitting?

(A) \$16.00 (B) \$19.00 (C) \$32.50 (D) \$13.50 (E) \$28.00

Source: 2006 Gauss Grade 7 #15

Primary Topics: Number Sense

Secondary Topics: Percentages | Rates

Answer: C

Solution:

Since Harry charges 50% more for each additional hour as he did for the previous hour, then he charges 1.5 or $\frac{3}{2}$ times as much as he did for the previous hour.
Harry charges \$4 for the first hour.
Harry then charges $\frac{3}{2} \times \$4 = \$6$ for the second hour.
Harry then charges $\frac{3}{2} \times \$6 = \$9$ for the third hour.
Harry then charges $\frac{3}{2} \times \$9 = \$\frac{27}{2} = \$13.50$ for the fourth hour.
Therefore, for 4 hours of babysitting, Harry would earn $\$4 + \$6 + \$9 + \$13.50 = \$32.50$.

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12. Keiko and Leah run on a track that is 150 m around. It takes Keiko 120 seconds to run 3 times around the track, and it takes Leah 160 seconds to run 5 times around the track. Who is the faster runner and at approximately what speed does she run?
(A) Keiko, 3.75 m/s (B) Keiko, 2.4 m/s (C) Leah, 3.3 m/s
(D) Leah, 4.69 m/s (E) Leah, 3.75 m/s

Source: 2006 Gauss Grade 7 #18

Primary Topics: Number Sense

Secondary Topics: Rates

Answer: D

Solution:

Solution 1
Since Keiko takes 120 seconds to run 3 times around the track, then it takes her $\frac{1}{3} \times 120 = 40$ seconds to run 1 time around the track.
Since Leah takes 160 seconds to run 5 times around the track, then it takes her $\frac{1}{5} \times 160 = 32$ seconds to run 1 time around the track.
Since Leah takes less time to run around the track than Keiko, then she is the faster runner.
Since Leah takes 32 seconds to run the 150 m around the track, then her speed is $\frac{150 \text{ m}}{32 \text{ s}} = 4.6875 \text{ m/s} \approx 4.69 \text{ m/s}$.
Therefore, Leah is the faster runner and her speed is approximately 4.69 m/s.

Solution 2
In 120 seconds, Keiko runs 3 times around the track, or $3 \times 150 = 450$ m in total. Therefore, her speed is $\frac{450 \text{ m}}{120 \text{ s}} = 3.75 \text{ m/s}$.
In 160 seconds, Leah runs 5 times around the track, or $5 \times 150 = 750$ m in total. Therefore, her speed is $\frac{750 \text{ m}}{160 \text{ s}} = 4.6875 \text{ m/s} \approx 4.69 \text{ m/s}$.
Since Leah's speed is larger, she is the faster runner and her speed is approximately 4.69 m/s.

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13. At a potluck, the ratio of vegetarians to non-vegetarians is $3 : 7$. If there are 21 vegetarians at the potluck, what is the *total* number of people at the potluck?
(A) 30 (B) 25 (C) 49 (D) 70 (E) 79

Source: 2007 Pascal Grade 9 #14

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios

Answer: D

Solution:

Since there were 21 vegetarians and the ratio of vegetarians to non-vegetarians in attendance is $3 : 7$, then there are $\frac{7}{3} \times 21 = 49$ non-vegetarians in attendance.
Therefore, the total number of attendees is $49 + 21 = 70$.

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14. A survey of 400 students at Cayley University found that the ratio of students who commute to students who live on campus is $3:2$. A survey of 600 students at Fermat University found that the ratio of students who commute to students who live on campus is $2:3$. When considering all the surveyed students from both universities, what is the ratio of students who commute to students who live on campus?

(A) $2:3$ (B) $12:13$ (C) $1:1$ (D) $6:5$ (E) $3:2$

Source: 2010 Pascal Grade 9 #13

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios

Answer: B

Solution:

Since the ratio of commuters to students who live on campus at Cayley University~is $3:2$, then $\frac{3}{3+2} = \frac{3}{5}$ of the students at Cayley University are commuters.
Thus, there are $\frac{3}{5}(400) = \frac{1200}{5} = 240$ commuters at Cayley University
Since the ratio of commuters to students who live on campus at Fermat University~is $2:3$, then $\frac{2}{2+3} = \frac{2}{5}$ of the students at Fermat University~are commuters.
Thus, there are $\frac{2}{5}(600) = \frac{1200}{5} = 240$ commuters at Fermat University.
There are $400 + 600 = 1000$ students in total at the two schools.
Of these, $240 + 240 = 480$ are commuters, and so the remaining $1000 - 480 = 520$ students live on campus.
Therefore, the overall ratio of commuters to students who live on campus is $480:520 = 48:52 = 12:13$.

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15. One soccer ball and one soccer shirt together cost \$100. Two soccer balls and three soccer shirts together cost \$262. What is the cost of one soccer ball?

(A) \$38 (B) \$50 (C) \$87.30 (D) \$45 (E) \$40

Source: 2016 Gauss Grade 8 #14

Primary Topics: Algebra and Equations

Secondary Topics: Rates

Answer: A

Solution:

One soccer ball and one soccer shirt together cost \$100.
So then two soccer balls and two soccer shirts together cost $2 \times \$100 = \$200$.
Since we are given that two soccer balls and three soccer shirts together cost \$262, then \$200 added to the cost of one soccer shirt is \$262.
Thus, the cost of one soccer shirt is $\$262 - \$200 = \$62$, and the cost of one soccer ball is $\$100 - \$62 = \$38$.

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16. Chris received a mark of 50% on a recent test. Chris answered 13 of the first 20 questions correctly. Chris also answered 25% of the remaining questions on the test correctly. If each question on the test was worth one mark, how many questions in total were on the test?
(A) 23 (B) 38 (C) 32 (D) 24 (E) 40

Source: 2016 Pascal Grade 9 #19

Primary Topics: Number Sense

Secondary Topics: Percentages

Answer: C

Solution:

Suppose that there were n questions on the test.
Since Chris received a mark of 50% on the test, then he answered $\frac{1}{2}n$ of the questions correctly.
We know that Chris answered 13 of the first 20 questions correctly and then 25% of the remaining questions.
Since the test has n questions, then after the first 20 questions, there are $n - 20$ questions.
Since Chris answered 25% of these $n - 20$ questions correctly, then Chris answered $\frac{1}{4}(n-20)$ of these questions correctly.
The total number of questions that Chris answered correctly can be expressed as $\frac{1}{2}n$ and also as $13 + \frac{1}{4}(n-20)$.
Therefore, $\frac{1}{2}n = 13 + \frac{1}{4}(n-20)$ and so $2n = 52 + (n-20)$, which gives $n = 32$.
(We can check that if $n = 32$, then Chris answers 13 of the first 20 and 3 of the remaining 12 questions correctly, for a total of 16 correct out of 32.)

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17. Brodie and Ryan are driving directly towards each other. Brodie is driving at a constant speed of 50 km/h. Ryan is driving at a constant speed of 40 km/h. If they are 120 km apart, how long will it take before they meet?
(A) 1 h 12 min (B) 1 h 25 min (C) 1 h 15 min (D) 1 h 33 min (E) 1 h 20 min

Source: 2017 Gauss Grade 8 #18

Primary Topics: Number Sense

Secondary Topics: Rates

Answer: E

Solution:

When Brodie and Ryan are driving directly towards each other at constant speeds of 50 km/h and 40 km/h respectively, then the distance between them is decreasing at a rate of $50 + 40 = 90$ km/h.

If Brodie and Ryan are 120 km apart and the distance between them is decreasing at 90 km/h, then they will meet after $\dfrac{120}{90}$ h or $\dfrac{4}{3}$ h or $1\frac{1}{3}$ h.

Since $\frac{1}{3}$ of an hour is $\frac{1}{3} \times 60 = 20$ minutes, then it will take Brodie and Ryan 1 h 20 min to meet.

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18. Jeff and Ursula each run 30 km. Ursula runs at a constant speed of 10 km/h. Jeff also runs at a constant speed. If Jeff's time to complete the 30 km is 1 hour less than Ursula's time to complete the 30 km, at what speed does Jeff run?
(A) 6 km/h (B) 11 km/h (C) 12 km/h (D) 15 km/h (E) 22.5 km/h

Source: 2017 Pascal Grade 9 #11

Primary Topics: Geometry and Measurement | Number Sense

Secondary Topics: Rates

Answer: D

Solution:

When Ursula runs 30 km at 10 km/h, it takes her $\frac{30\text{ km}}{10\text{ km/h}} = 3\text{ h}$.
This means that Jeff completes the same distance in $3\text{ h} - 1\text{ h} = 2\text{ h}$.
Therefore, Jeff's constant speed is $\frac{30\text{ km}}{2\text{ h}} = 15\text{ km/h}$.

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19. Tobias downloads m apps. Each app costs \$2.00 plus 10% tax. He spends \$52.80 in total on these m apps. What is the value of m?
(A) 20 (B) 22 (C) 18 (D) 24 (E) 26

Source: 2017 Pascal Grade 9 #14

Primary Topics: Algebra and Equations

Secondary Topics: Equations Solving | Percentages

Answer: D

Solution:

Since the tax rate is 10%, then the tax on each \$2.00 app is $\$2.00 \times \frac{10}{100} = \$0.20$.
Therefore, including tax, each app costs $\$2.00 + \$0.20 = \$2.20$.
Since Tobias spends \$52.80 on apps, he downloads $\frac{\$52.80}{\$2.20} = 24$ apps.
Therefore, $m = 24$.

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20. A bag contains only green, yellow and red marbles. The ratio of green marbles to yellow marbles to red marbles in the bag is $3:4:2$. If 63 of the marbles in the bag are *not* red, the number of red marbles in the bag is
(A) 14 (B) 18 (C) 27 (D) 36 (E) 81

Source: 2020 Cayley Grade 10 #15

Primary Topics: Algebra and Equations

Secondary Topics: Fractions/Ratios

Answer: B

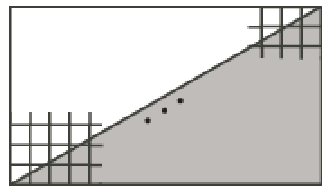
Solution:

Since the ratio of green marbles to yellow marbles to red marbles is $3:4:2$, then we can let the numbers of green, yellow and red marbles be $3n$, $4n$ and $2n$ for some positive integer n.

Since 63 of the marbles in the bag are not red, then the sum of the number of green marbles and the number of yellow marbles in the bag is 63.

Thus, $3n+4n=63$ and so $7n=63$ or $n=9$, which means that the number of red marbles in the bag is $2n=2\times 9=18$.

21. Unit squares are arranged to form a rectangular grid that is m units wide and n units tall, where m and n are positive integers with $2n < m < 3n$. The region below one of the diagonals of the rectangle is shaded as shown. For certain pairs m and n, there is a unit square in the grid that is not completely shaded but whose shaded area is greater than 0.999. The smallest possible value of mn for which this is true satisfies



(A) $496 \leq mn \leq 500$ (B) $501 \leq mn \leq 505$ (C) $506 \leq mn \leq 510$
(D) $511 \leq mn \leq 515$ (E) $516 \leq mn \leq 520$

Source: 2009 Cayley Grade 10 #25

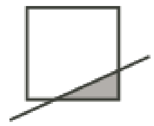
Primary Topics: Geometry and Measurement

Secondary Topics: Area | Fractions/Ratios

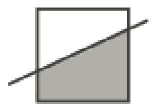
Answer: C

Solution:

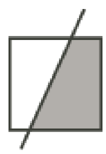
We add coordinates to the diagram, with the bottom left corner at $(0,0)$, the bottom right at $(m,0)$, the top right at (m,n), and the top left at $(0,n)$.
Thus, the slope of the diagonal is $\dfrac{n}{m}$.
This tells us that the equation of the diagonal is $y=\dfrac{n}{m}x$.
Since $2n<m<3n$, then $\dfrac{1}{3}<\dfrac{n}{m}<\dfrac{1}{2}$; that is, the slope is between $\frac{1}{3}$ and $\frac{1}{2}$.
There are three possible configurations of shading in these partially shaded squares: A small triangle is shaded, while the rest is unshaded:



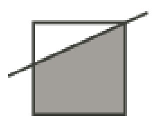
Here, the maximum possible length of the base is 1 and so the maximum possible height is when the slope is as large as possible, so is $\frac{1}{2}$.
Thus, in this case, the maximum shaded area is $\frac{1}{2}(1)(\frac{1}{2})=\frac{1}{4}$.
Since we want a shaded area of more than 0.999, then this is not the case we need. A trapezoid is shaded and a trapezoid is unshaded:



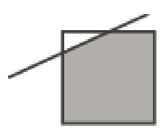
(Note that since the slope is less than 1, then the case



is not possible.)
Consider the unshaded area. For the shaded area to be more than 0.999, the unshaded area is less than 0.001.
But the unshaded trapezoid is at least as big as the triangle that is cut off when the diagonal passes through a vertex:



Such a triangle has base 1 and height at least $\frac{1}{3}$ (since the slope is at least $\frac{1}{3}$).
Thus, the area of such a trapezoid is at least $\frac{1}{2}(1)(\frac{1}{3})=\frac{1}{6}$ and so cannot be less than 0.001. A triangle is unshaded:



It is this last case upon which we need to focus.
Suppose that the coordinates of the top left corner of such a unit square are (p,q).
The point where the diagonal ($y=\dfrac{n}{m}x$) crosses the top edge of the square ($y=q$) has coordinates $\left(\dfrac{m}{n}q,q\right)$, since if $y=q$, then $q=\dfrac{n}{m}x$ gives $x=\dfrac{m}{n}q$.
Similarly, the point where the diagonal crosses the left edge ($x=p$) of the square has coordinates $\left(p,\dfrac{n}{m}p\right)$.
Thus, the triangle has (horizontal) base of length $\dfrac{m}{n}q-p$ and (vertical) height of length $q-\dfrac{n}{m}p$.
We also know that neither the base nor the height is 0, since there is some unshaded area.
Since the area of the unshaded triangle is less than 0.001, then

$$\begin{array}{rclcl} 0 & < & \frac{1}{2}\left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.001 \\ 0 & < & \left(\frac{m}{n}q - p\right)\left(q - \frac{n}{m}p\right) & < & 0.002 \\ 0 & < & (mq - pn)(mq - pn) & < & 0.002mn \qquad \text{(multiplying by } mn\text{)} \\ 0 & < & 500(mq - pn)^2 & < & mn \end{array}$$

Now m, n, p and q are integers and $mq - pn$ is not zero. In fact, $mq - pn = n\left(\frac{m}{n}q - p\right) > 0$.
Thus, $(mq - pn)^2 \geq 1$ because m, q, p, n are all integers and $(mq - pn)^2 > 0$.
Thus, $mn > 500(1) = 500$.
Note that if $(mq - pn)^2 > 1$, then mn would be much bigger.
So, since we want the smallest value of mn, we try to see if we can find a solution with $(mq - pn)^2 = 1$.

So we need to try to find m and n with $2n < m < 3n$, with the product mn as close to 500 as possible, and so that we can also find p and q with $mq - pn = 1$.
We consider the restriction that $2n

- $501 = 3(167)$ and 167 is prime, so this is not possible
- $502 = 2(251)$ and 251 is prime, so this is not possible
- 503 is prime, so this is not possible
- $504 = 8(7)(9)$ so we can choose $n = 14$ and $m = 36$ (this is the only such way)
- $505 = 5(101)$ and 101 is prime, so this is not possible
- $506 = 11(2)(23)$ which cannot be written in this way
- $507 = 3(13)(13)$ which cannot be written in this way
- $508 = 4(127)$ and 127 is prime, so this is not possible
- 509 which is prime, so this is not possible
- $510 = 2(3)(5)(17)$, so we can choose $n = 15$ and $m = 34$ (this is only such way)

This gives two possible pairs m and n to consider so far. If one of them works, then this pair will give the smallest possible value of mn.
In order to verify if one of these works, we do need to determine if we can find an appropriate p and q.
Consider $n = 14$ and $m = 36$. In this case, we want to find integers p and q with $36q - 14p = 1$. This is not possible since the left side is even and the right side is odd.
Consider $n = 15$ and $m = 34$. In this case, we want to find integers p and q with $34q - 15p = 1$.
The integers $q = 4$ and $p = 9$ satisfy this equation.
Therefore, $(m, n) = (34, 15)$ is a pair with the smallest possible value of mn that satisfies the given conditions, and so $mn = 510$.

22. Dolly, Molly and Polly each can walk at 6 km/h. Their one motorcycle, which travels at 90 km/h, can accommodate at most two of them at once (and cannot drive by itself!). Let t hours be the time taken for all three of them to reach a point 135 km away. Ignoring the time required to start, stop or change directions, what is true about the smallest possible value of t?

(A) $t < 3.9$ (B) $3.9 \le t < 4.1$ (C) $4.1 \le t < 4.3$ (D) $4.3 \le t < 4.5$
(E) $t \ge 4.5$

Source: 2011 Cayley Grade 10 #24

Primary Topics: Algebra and Equations

Secondary Topics: Inequalities | Rates

Answer: A

Solution:

First, we note that the three people are interchangeable in this problem, so it does not matter who rides and who walks at any given moment. We abbreviate the three people as D, M and P.
We call their starting point A and their ending point B.
Here is a strategy where all three people are moving at all times and all three arrive at B at the same time:
D and M get on the motorcycle while P walks.
D and M ride the motorcycle to a point Y before B.
D drops off M and rides back while P and M walk toward B.
D meets P at point X.
D picks up P and they drive back to B meeting M at B.
Point Y is chosen so that D, M and P arrive at B at the same time.
Suppose that the distance from A to X is a km, from X to Y is d km, and the distance from Y to B is b~km.

In the time that it takes P to walk from A to X at 6 km/h, D rides from A to Y and back to X at 90 km/h.
The distance from A to X is a km.
The distance from A to Y and back to X is $a+d+d=a+2d$ km.
Since the time taken by P and by D is equal, then $\frac{a}{6}=\frac{a+2d}{90}$ or $15a=a+2d$ or $7a=d$.

In the time that it takes M to walk from Y to B at 6 km/h, D rides from Y to X and back to B at 90 km/h.
The distance from Y to B is b km, and the distance from Y to X and back to B is $d+d+b=b+2d$ km.
Since the time taken by M and by D is equal, then $\frac{b}{6}=\frac{b+2d}{90}$ or $15b=b+2d$ or $7b=d$.

Therefore, $d=7a=7b$, and so we can write $d=7a$ and $b=a$.
Thus, the total distance from A to B is $a+d+b=a+7a+a=9a$ km.
However, we know that this total distance is 135 km, so $9a=135$ or $a=15$.
Finally, D rides from A to Y to X to B, a total distance of $(a+7a)+7a+(7a+a)=23a$ km.
Since $a=15$ km and D rides at 90 km/h, then the total time taken for this strategy is
$\frac{23\times15}{90}=\frac{23}{6}\approx 3.83$ h.
Since we have a strategy that takes 3.83 h, then the smallest possible time is no more than 3.83~h.
Can you explain why this is actually the smallest possible time?

If we didn't think of this strategy, another strategy that we might try would be:
D and M get on the motorcycle while P walks.
D and M ride the motorcycle to B.
D drops off M at B and rides back to meet P, who is still walking.
D picks up P and they drive back to B. (M rests at B.)
This strategy actually takes 4.125 h, which is longer than the strategy shown above, since M is actually sitting still for some of the time.

23. Angie has a jar that contains 2 red marbles, 2 blue marbles, and no other marbles. She randomly draws 2 marbles from the jar. If the marbles are the same colour, she discards one and puts the other back into the jar. If the marbles are different colours, she discards the red marble and puts the blue marble back into the jar. She repeats this process a total of three times. What is the probability that the remaining marble is red?

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{2}{3}$ (D) $\frac{1}{3}$ (E) 0

Source: 2016 Gauss Grade 7 #23

Primary Topics: Counting and Probability

Secondary Topics: Probability | Fractions/Ratios

Answer: E

Solution:

Solution 1:

Let the letter R represent a red marble, and the letter B represent a blue marble.
On her first draw, Angie may draw RR, RB or BB.

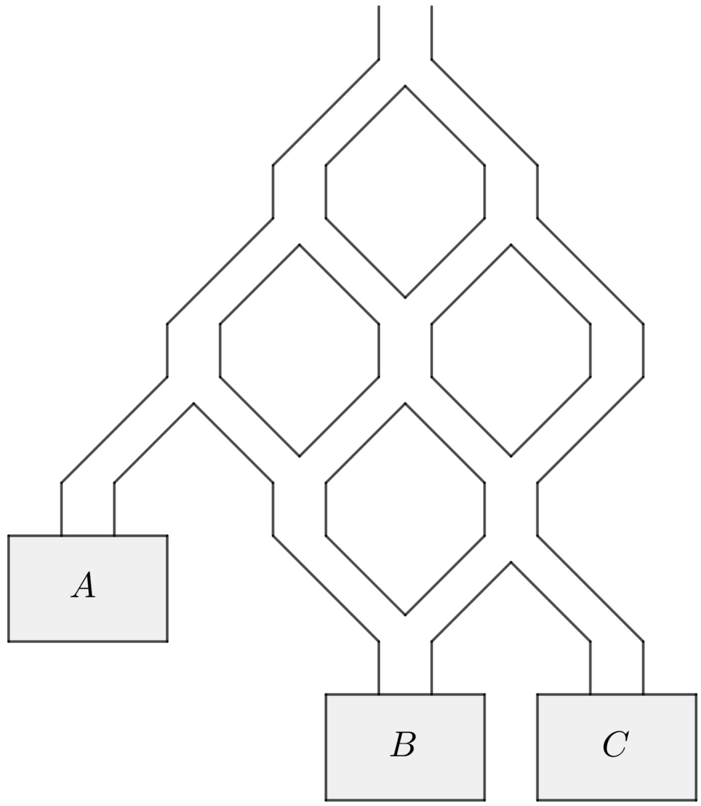
Case 1: Angie draws RR or RB on her first draw
If Angie draws RR or RB on her first draw, then she discards the R and the three remaining marbles in the jar are RBB.
On her second draw, Angie may draw RB or BB.
If she draws RB, then she discards the R and the two remaining marbles in the jar are BB.
Since there are no red marbles remaining, it is not possible for the final marble to be red in this case.
If on her second draw Angie instead draws BB, then she discards a B and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
Again in this case it is not possible for the final marble to be red.
Thus, if Angie draws RR or RB on her first draw, the probability that the final marble is red is zero.

Case 2: Angie draws BB on her first draw
If Angie draws BB on her first draw, then she discards a B and the three remaining marbles in the jar are RRB.
On her second draw, Angie may draw RR or RB.
If she draws RR or RB, then she discards one R and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
In this case it is not possible for the final marble to be red.
Thus, if Angie draws BB on her first draw, the probability that the final marble is red is zero.
Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

Solution 2:

Let the letter R represent a red marble, and the letter B represent a blue marble.
If the final remaining marble is R, then the last two marbles must include at least one R.
That is, the last two marbles must be RB or RR.
If the last two marbles are RB, then when they are drawn from the jar, the R is discarded and the B would remain.
Thus it is not possible for the final marble to be R if the final two marbles are RB.
So the final remaining marble is R only if the final two marbles are RR.
If the final two marbles are RR, then the last three marbles are BRR (since there are only two Rs in the jar at the beginning).
However, if the final three marbles are BRR, then when Angie draws two of these marbles from the jar, at least one of the marbles must be R and therefore one R will be discarded leaving BR as the final two marbles in the jar.
That is, it is not possible for the final two marbles in the jar to be RR.
The only possibility that the final remaining marble is R occurs when the final two marbles are RR, but this is not possible.
Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

24. A network of pathways lead from a single opening to three bins, labelled A, B, C as shown. If a ball is dropped into the opening, it will follow a path and land in one of the bins. Every time a path splits, it is equally likely for the ball to follow either of the downward paths.



Ellen drops two balls, one after the other, into the opening. What is the probability that the two balls land in different bins?

(A) $\frac{17}{32}$ (B) $\frac{27}{50}$ (C) $\frac{25}{64}$ (D) $\frac{1}{3}$ (E) $\frac{15}{32}$

Source: 2024 Gauss Grade 8 #24

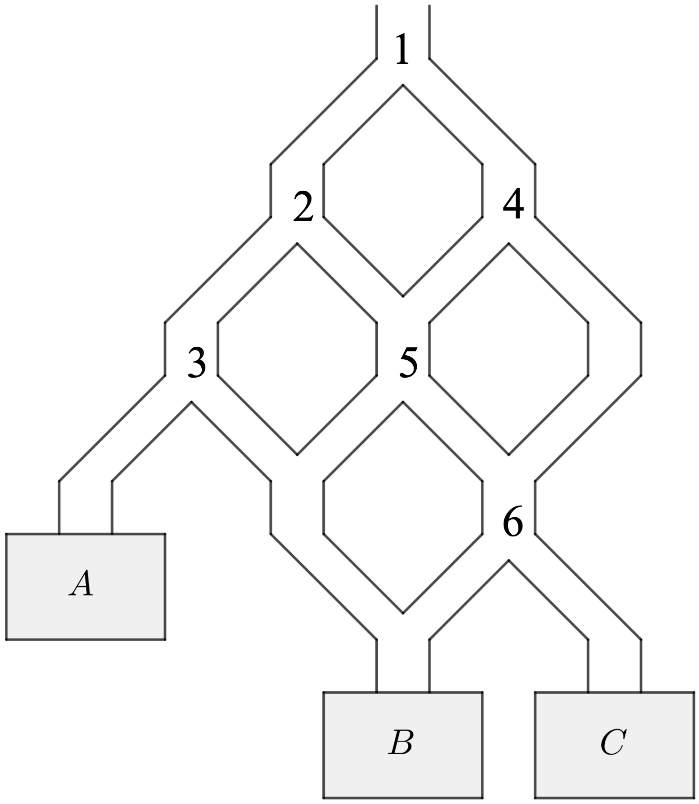
Primary Topics: Counting and Probability

Secondary Topics: Probability | Fractions/Ratios

Answer: A

Solution:

There are 6 different locations at which the path splits, and we label these splits 1 to 6, as shown.



We begin by determining the probability that a ball lands in the bin labelled A.
There is exactly one path that leads to bin A.
This path travels downward to the left at each of the three splits labelled 1, 2 and 3.
At each of these splits, the probability that a ball travels to the left is $\frac{1}{2}$, and so the probability that a ball lands in bin A is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Next, we determine the probability that a ball lands in the bin labelled C.
There are exactly three paths that lead to bin C.
One of these paths travels downward to the right at each of the three splits labelled 1, 4 and 6.
Thus, the probability that a ball lands in bin C by following this path is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
A second path to bin C travels downward to the right at split 1, to the left at split 4, to the right at split 5, and to the right at split 6.
The probability that a ball follows this path is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.
The third and final path to bin C travels left at split 1, and to the right at each of the three splits 2, 5 and 6.
The probability that a ball follows this path is also $\frac{1}{16}$.
The probability that a ball lands in bin C is the sum of the probabilities of travelling each of these three paths or

$$\frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{2+1+1}{16} = \frac{4}{16} = \frac{1}{4}$$

Finally, we determine the probability that a ball lands in bin B.
There are six different paths that lead to bin B, and we could determine the probability that a ball follows each of these just as we did for bins A and C.
However, it is more efficient to recognize that a ball must land in one of the three bins, and thus the probability that it lands in bin B is 1 minus the probability that it lands in bin A minus the probability that it lands in bin C, or

$$1 - \frac{1}{8} - \frac{1}{4} = \frac{8-1-2}{8} = \frac{5}{8}$$

The probability that the two balls land in different bins is equal to 1 minus the probability that the two balls land in the same bin.
The probability that a ball lands in bin A is $\frac{1}{8}$, and so the probability that two balls land in bin A is $\frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$.
The probability that a ball lands in bin C is $\frac{1}{4}$, and so the probability that two balls land in bin C is $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.
The probability that a ball lands in bin B is $\frac{5}{8}$, and so the probability that two balls land in bin B is $\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$.
Therefore, the probability that the two balls land in different bins is equal to

$$1 - \frac{1}{64} - \frac{1}{16} - \frac{25}{64} = \frac{64-1-4-25}{64} = \frac{34}{64} = \frac{17}{32}$$

From split 1, moving to the left leads to split 2 and moving to the right leads to split 4.
From 2, left leads to 3 and right leads to 5.
From 4, left leads to 5 and right leads to 6.
From 3, left leads to bin A and right leads to bin B.
From 5, left leads to bin B and right leads to split 6.
From 6, left leads to bin B and right leads to bin C.

25. Consider positive integers $a \leq b \leq c \leq d \leq e$. There are N lists a, b, c, d, e with a mean of 2023 and a median of 2023, in which the integer 2023 appears more than once, and in which no other integer appears more than once. What is the sum of the digits of N?

Source: 2023 Pascal Grade 9 #25

Primary Topics: Algebra and Equations

Secondary Topics: Inequalities | Averages | Decimals

Answer: 28

Solution:

Since the median of the list a, b, c, d, e is 2023 and $a \leq b \leq c \leq d \leq e$, then $c = 2023$. Since 2023 appears more than once in the list, then it appears 5, 4, 3, or 2 times.
Case 1: 2023 appears 5 times.
Here, the list is 2023, 2023, 2023, 2023, 2023. There is 1 such list.
Case 2: 2023 appears 4 times.
Here, the list would be 2023, 2023, 2023, 2023, x where x is either less than or greater than 2023. Since the mean of the list is 2023, the sum of the numbers in the list is 5×2023, which means that $x = 5 \times 2023 - 4 \times 2023 = 2023$, which is a contradiction. There are 0 lists in this case.
Case 3: 2023 appears 3 times.
Here, the list is a, b, 2023, 2023, 2023 (with $a < b < 2023$) or a, 2023, 2023, 2023, e (with $a < 2023 < e$), or 2023, 2023, 2023, d, e (with $2023 < d < e$). In the first case, the mean of the list is less than 2023, since the sum of the numbers will be less than 5×2023. In the third case, the mean of the list is greater than 2023, since the sum of the numbers will be greater than 5×2023. So we need to consider the list a, 2023, 2023, 2023, e with $a < 2023 < e$. Since the mean of this list is 2023, then the sum of the five numbers is 5×2023, which means that $a + e = 2 \times 2023$. Since a is a positive integer, then $1 \leq a \leq 2022$. For each such value of a, there is a corresponding value of e equal to $4046 - a$, which is indeed greater than 2023. Since there are 2022 choices for a, there are 2022 lists in this case.
Case 4A: 2023 appears 2 times; $c = d = 2023$.
(We note that if 2023 appears 2 times, then since $c = 2023$ and $a \leq b \leq c \leq d \leq e$, we either have $c = d = 2023$ or $b = c = 2023$.) Here, the list is a, b, 2023, 2023, e with $1 \leq a < b < 2023 < e$. This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean. For this to be the case, the sum of its numbers equals 5×2023, which means that $a + b + e = 3 \times 2023 = 6069$. Every pair of values for a and b with $1 \leq a < b < 2023$ will give such a list by defining $e = 6069 - a - b$. (We note that since $a<b<2023$ we will indeed have $e > 2023$.) If $a = 1$, there are 2021 possible values for b, namely $2 \leq b \leq 2022$. If $a = 2$, there are 2020 possible values for b, namely $3 \leq b \leq 2022$. Each time we increase a by 1, there will be 1 fewer possible value for b, until $a = 2021$ and $b = 2022$ (only one value). Therefore, the number of pairs of values for a and b in this case is

$$2021 + 2020 + \cdots + 2 + 1 = \tfrac{1}{2} \times 2021 \times 2022 = 2021 \times 1011$$

This is also the number of lists in this case.
Case 4B: 2023 appears 2 times; $b = c = 2023$.
Here, the list is a, 2023, 2023, d, e with $1 \leq a < 2023 < d < e$. This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean. For this to be the case, the sum of its numbers equals 5×2023, which means that $a + d + e = 3 \times 2023 = 6069$. If $d = 2024$, then $a + e = 4045$. Since $1 \leq a \leq 2022$ and $2025 \leq e$, we could have $e = 2025$ and $a = 2020$, or $e = 2026$ and $a = 1019$, and so on. There are 2020 such pairs, since once a reaches 1, there are no more possibilities. If $d = 2025$, then $a + e = 4044$. Since $1 \leq a \leq 2022$ and $2026 \leq e$, we could have $e = 2026$ and $a = 2018$, or $e = 2027$ and $a = 1017$, and so on. There are 2018 such pairs. As d increases successively by 1, the sum $a + e$ decreases by 1 and the minimum value for e increases by 1, which means that the maximum value for a decreases by 2, which means that the number of pairs of values for a and e decreases by 2. This continues until we reach $d = 3033$ at which point there are 2 pairs for a and e. Therefore, the number of pairs of values for a and e in this case is

$$2020 + 2018 + 2016 + \cdots + 4 + 2$$

which is equal to

$$2 \times (1 + 2 + \cdots + 1008 + 1009 + 1010)$$

which is in turn equal to $2 \times \frac{1}{2} \times 1010 \times 1011$ which equals1010×1011.
Combining all of the cases, the total number of lists a, b, c, d, e is

$$N = 1 + 2022 + 2021 \times 1011 + 1010 \times 1011 = 1 + 1011 \times (2 + 2021 + 1010) = 1 + 1011 \times 3033$$

and so $N = 3\,066\,364$. The sum of the digits of N is $3 + 0 + 6 + 6 + 3 + 6 + 4$ or 28.